Local & Global Errors : Stability

Want to solve

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

Recall : multistep methods are of the form

$$\left.\begin{aligned} a_k w_n + a_{k-1} w_{n-1} + \dots + a_0 w_{n-k} \\ = h\left[b_k f_n + b_{k-1} f_{n-1} + \dots + b_0 f_{n-k}\right] \end{aligned}\right\} (*)$$

($f_n \to f(t_n, w_n)$)

$b_k = 0 \Rightarrow$ explicit method

$b_k \neq 0 \Rightarrow$ implicit method (why?)

The multistep method $(*)$ is said to be convergent if :

denoting by $w_n(h,t)$ the solution to $(*)$, we have

$$\lim_{h \to 0} w(h,t) = y(t)$$

($y(t)$: true solution)

for all $t \in [t_0, t_m]$ provided

$$\lim_{h \to 0} w(h, t_0 + nh) = w_0 \quad (0 \le n < k)$$

and provided f satisfies the cond'ns of the Lipschitz theorem for exist. & uniqueness.

$\Leftrightarrow$ $\max_{n \in \{0, \dots, \frac{t_m - t_0}{h}\}} |w_n - y_n| \to 0$ as $h \to 0$

Stability and Consistency

Define the polynomials (associated with (*)) ρ, σ:

$$\rho(z) = a_k z^k + a_{k-1} z^{k-1} + \dots + a_0$$

$$\sigma(z) = b_k z^k + b_{k-1} z^{k-1} + \dots + b_0$$

(small changes to initial cond'n
⇒ small change to approx.)

We say the method is **stable** if all roots of p satisfy $|z| \le 1$ (in the unit disk) and all roots with $|z| = 1$ are simple (not repeated).

We say the method is **consistent** if $p(1) = 0$ & $p'(1) = q(1)$

↳ (difference eq'n $\xrightarrow[as\ h \to 0]{}$ differential eq'n)

Theorem (Convergence)

(Method is stable & consistent)

$\iff$ (Method is convergent)

Proof (See book for ($\Leftarrow$))

Example

Consider the milne method

$$x_n - x_{n-2} = h\left[\frac{1}{3}f_n + \frac{4}{3}f_{n-1} + \frac{1}{3}f_{n-2}\right]$$

Then $p(z) = z^2 - 1$

$$q(z) = \frac{1}{3}z^2 + \frac{4}{3}z + \frac{1}{3}$$

$p(z) = 0 \Rightarrow z = \pm 1$ (simple roots)

so the method is stable

and since $p(1) = 0$ & $p'(1) = q(1)$

$= 2$

& $q(1) = \frac{1}{3}(1)^2 + \frac{4}{3}(1) + \frac{1}{3} = 2$

$= p(1)$

then it is consistent, so

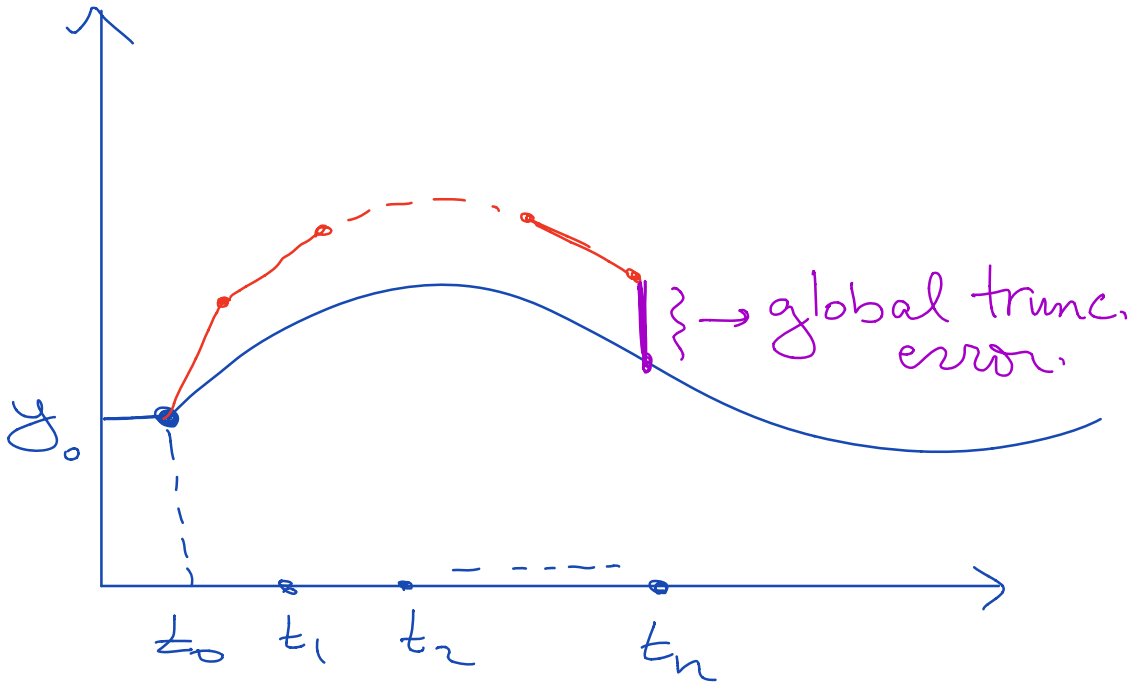
by the theorem it is convergent

More on Local Truncation Error

Recall: Local Truncation error is $w_n - y(t_n)$ assuming all previous estimates $w_{n-1}, w_{n-2}, \ldots$ are correct.

Theorem: For a multistep method (*) of order m:
if $y \in C^{m+2}$, and $\frac{\partial f}{\partial y}$ is cont's
then

$$w_n - y(t_n) = O(h^{m+1})$$



Theorem:

Local Truncation error $= O(h^{m+1})$

$\Rightarrow$ Global Truncation error $= O(h^m)$

Systems & higher order ODE's

Example: predator-prey models

$$\begin{cases} y_1' = f_1(t, y_1, y_2) \\ y_2' = f_2(t, y_1, y_2) \end{cases}$$

Given some initial conditions

$$y_1(t_0) = \alpha, \quad y_2(t_0) = \beta$$

this becomes an IVP.

More generally:

$$\begin{cases} y_1' = f_1(t, y_1, \dots, y_n) \\ y_2' = f_2(t, y_1, \dots, y_n) \\ \vdots \\ y_n' = f_n(t, y_1, \dots, y_n) \end{cases}$$

We can write this in vector notation:

$$\begin{cases} Y' = F(t, Y) \\ \qquad \hookrightarrow \text{function } \mathbb{R}^{n+1} \to \mathbb{R}^n \\ Y(t_0) = Y_0 \\ \qquad \hookrightarrow \in \mathbb{R}^n \end{cases}$$

($Y \in \mathbb{R}^n$)

Fact: Can convert a higher order ODE to a system of first order ODE's

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

$$y \leftrightarrow x_1, \quad y' \leftrightarrow x_2, \quad y^{(n-1)} \leftrightarrow x_n$$

$$\Leftrightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \end{cases}$$

$$\begin{cases} x_1' = f(t, x_1, \dots, x_n) \\ \vdots \\ x_n' = f(t, x_1, \dots, x_n) \end{cases}$$

Example: Convert the IVP below to a system of first-order ODE's with initial values

$$\begin{cases} (\sin t)y''' + (\cos ty) + \sin(t^2 + y'') + (y')^3 \\ \quad = \log(t) \\ y(2) = 7 \\ y'(2) = 3, \quad y''(2) = -4 \end{cases}$$

Soln: Introduce x_1, x_2, x_3:

Let $\left.\begin{aligned} x_1 &= y \\ x_2 &= x_1' \\ x_3 &= x_2' \end{aligned}\right\}$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = \left[\log t - x_2^3 - \sin(t^2 + x_3) - \cos(tx_1)\right] / \sin t \end{cases}$$

with $X(2) = (7, 3, -4)$

Example

Convert the system below into a system of first order ODE's

$$\begin{cases} (x')^2 + te^y + y' = x' - x \\ y'y'' - \cos(xy) + \sin(tx'y) = t \end{cases}$$

Soln: Let $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = (x_2 - x_1 - te^{x_3} - x_4)^{1/2} \\ x_3' = x_4 \\ x_4' = \dfrac{(x_1 + \cos(x_1 x_3) - \sin(t x_2 x_3))}{x_4} \end{cases}$$

So we've seen that higher order ODE's, and systems of higher order ODE's, can be rewritten as systems of 1st order ODE's. How do we solve them?

Solving systems of 1st order ODE's

- Taylor series method

We write a taylor series approx. for each variable:

$$x_i(t+h) = x_i(t) + x_i'(t)h + \frac{x_i''(t)h^2}{2} + \cdots + \frac{x_i^{(n)}(t)h^n}{n!} + \text{error}$$

$\Leftrightarrow$ In vector notation

$$X(t+h) = X(t) + X'(t)h + \frac{X''(t)h^2}{2} + \cdots + \frac{X^{(n)}(t)h^n}{n!} + \text{error}$$

Recall: one variable case, Taylor method — idea behind

$$y' = f(t,y)$$

$$y(t+h) = y(t) + y'(t)h + y''(t)\frac{h^2}{2} + \cdots + y^{(m)}\frac{h^m}{m!} + \text{error}$$

$$\Rightarrow w_{n+1} = w_n + h\left[f_n + f_n'\frac{h}{2} + \cdots + \frac{f_n^{(m)}h^m}{m!}\right]$$

$f(t_n, w_n)$ — Chain rule!

Same idea for systems!

Example:

$$\begin{cases} x' = x + y^2 - t^3 =: f \\ y' = y + x^3 + \cos t =: g \\ x(1) = 3 \\ y(1) = 1 \end{cases}$$

Use $h = 0.1$, and write the second order taylor method

Soln: We'll need x', y' (~) x'', y''

$$x'' = x' + 2yy' - 3t^2 =: f'$$
$$y'' = y' + 3x^2x' - \sin t =: g'$$

Let w_n be the iterates associated with x & v_n be the iterates associated with y. Then with

$t_0 = 1$, $h = 0.1$, $t_n = t_0 + nh$

$$w_0 = 3$$
$$v_0 = 1$$
$$w_{n+1} = w_n + h\left[f_n + \frac{hf_n'}{2}\right]$$
$$v_{n+1} = v_n + h\left[g_n + \frac{hg_n'}{2}\right]$$

where $f_n = w_n + v_n^2 - t_n^3$

$$g_n = v_n + w_n^3 + \cos(t_n)$$
$$f_n' = f_n + 2v_ng_n - 3t_n^2$$
$$g_n' = g_n + 3w_n^2 f_n - \sin t_n$$

Other methods we studied for first order IVP's can also be generalized.

Example: RK4 to solve

$$X' = F(X)$$

Here $X = (x_0, x_1, \ldots, x_n)^T$

$x_0 \downarrow t$, $x_1 \downarrow x_1$, $\ldots$, $x_n \downarrow x_n$

& the first eq'n is $x_0' = 1$

$$X(t+h) = X(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

with

$$\begin{cases} F_1 = h\,F(X) \\ F_2 = h\,F(X + \frac{1}{2}F_1) \\ F_3 = h\,F(X + \frac{1}{2}F_2) \end{cases}$$

($F_4 = h F(X + F_3)$

(Looks very similar to 1 variable case but implementation requires care).

- Multi-step methods can also be generalized similarly.